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Relationships between various characterizations of wave tails

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Abstract. One can define several properties of wave equations that correspond to the absence of tails in their solutions, the most common, by far, being Huygens' principle. Not all of these definitions are equivalent, although they are sometimes assumed to be. We analyse this issue in detail for linear scalar waves, establishing some relationships between the various properties. Huygens' principle is almost always equivalent to the characteristic propagation property and in two spacetime dimensions the latter is equivalent to the zeroth-order progressing-wave propagation property. Higher-order progressing waves, in general, do have tails and do not seem to admit a simple physical characterization, but they are nevertheless useful because of their close association with exactly solvable two-dimensional equations.

1. Introduction

The question as to whether a propagating wave leaves a tail behind it is one that has interested many physicists and mathematicians for many years and has found applications from the first studies in light propagation [1] to the theory behind the proposed experiments to detect gravitational waves [2, 3]. Intuitively, a wave tail can be described as follows. Suppose that, at some time t, an instantaneous pulse of a field ϕ is produced at a point labelled by spatial coordinates $\{x^i\}$. This pulse originates a wave front propagating outward with some speed that will be detected by an observer sitting at some point $\{x'^i\}$ at a time $t' > t$. We say that the wave has a *tail* or *wake* if such an observer continues to detect a non-vanishing field even after the wave front has passed, i.e. at times greater than t'. In this case, we speak of diffusive, as opposed to sharp, propagation.

One can roughly identify three distinct possible reasons for the occurrence of tails.

(i) Mass-like terms in the wave equation. For example, the Klein–Gordon equation

$$\Box\phi - \mu^2\phi = 0 \tag{1.1}$$

in four-dimensional Minkowski spacetime has tails when $\mu \neq 0$, but not when $\mu = 0$. This feature can be seen as the wave-mechanical counterpart of the fact that massive particles move slower than the speed of light.

(ii) Dimensionality of spacetime. For example, the massless Klein–Gordon equation in m-dimensional Minkowski spacetime has tails for odd values of m, but not when m is even, with the exception of $m = 2$ [4–7].

(iii) Backscattering off potentials and/or spacetime curvature [3, 8, 9]. This is clearly the most interesting tail-production mechanism from a physical viewpoint.

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The study of the occurrence of tails, as well as of their implications, is certainly of great interest and importance, but relies on the possibility of using precise definitions. The heuristic characterization given above is evidently too loose for this purpose, but it can be easily formalized to obtain a tail-free property. For linear wave equations, however, a no-tails condition is better expressed in terms of the Green function and corresponds to what is usually called the Huygens principle. Furthermore, still other definitions (e.g. characteristic propagation property, progressing-wave propagation property) can be found in the literature. These characterizations are not all equivalent, although they are sometimes (explicitly or implicitly) assumed to be. It is the purpose of this paper to explore this issue in detail, by establishing the relationships between them and clarifying their meaning and domain of applicability.

If the wave equation is inhomogeneous (which corresponds, physically, to the presence of sources), the formulation of no-tails properties is a delicate task because of the need to capture the notion of sharp propagation when the field due to the sources is superposed to the one arising purely from boundary conditions. The situation is even worse for nonlinear waves. For these reasons, we shall restrict ourselves here to considering an arbitrary linear second-order homogeneous hyperbolic partial differential equation, which can always be written in the form

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi + h^\mu\partial_\mu\phi + K\phi = 0 \tag{1.2}$$

where $g^{\mu\nu}$, h^μ and K are suitable real functions of m variables $\{x^\mu\}$, which we can think of as coordinates over an m-dimensional differentiable manifold $\mathcal{M}$; without loss of generality, we can assume that $g^{\mu\nu} = g^{\nu\mu}$ so that these functions can be thought of as the components of the inverse of a Lorentzian metric g_{ab} on $\mathcal{M}$ given, in the coordinates $\{x^\mu\}$, by the inverse of the matrix $g^{\mu\nu}$; this metric, however, may not be the one normally used to determine intervals and causal relations (for a physical example in which it is not, see [10]). The metric g_{ab} then defines a Riemannian connection ∇ that allows us to rewrite (1.2) equivalently as

$$g^{ab}\nabla_a\nabla_b\phi + H^a\nabla_a\phi + K\phi = 0 \tag{1.3}$$

where H^a is uniquely determined by h^μ and $g^{\mu\nu}$ (specifically, $H^\mu = h^\mu - \partial_\nu g^{\mu\nu} - \partial^\mu \ln\sqrt{-g}$, where g is the determinant of $g_{\mu\nu}$). The form of (1.2) and (1.3) is sufficiently general to encompass many wave equations of interest in mathematical physics.

Although physically one may wish to discuss the appearance of tails in waves emitted by very distant sources, the phenomenon is essentially a local one. Therefore, it will not represent a loss in generality to study wave propagation close to the hypersurfaces where data are given, and we can assume that $\mathcal{M}$ is globally hyperbolic and normal. A more general manifold can always be covered by a collection of regions with these properties and tails will develop iff they already do inside some of those smaller domains [11]. In such a manifold, a very powerful tool for studying the general properties of linear differential equations and their solutions are the Green functions [6, 12]. A Green function $G(x, x')$ for (1.3) is defined by

$$(g^{ab}\nabla_a\nabla_b - H^a\nabla_a + K - \nabla_a H^a)G(x, x') = -\delta(x, x') \tag{1.4}$$

and by appropriate boundary conditions; in (1.4), $\delta(x, x')$ is the delta function on $\mathcal{M}$, such that for each test function f

$$\int \mathrm{d}^m x' \sqrt{-g(x')} f(x')\delta(x, x') = f(x). \tag{1.5}$$

It should be noted that the differential operator acting on the first variable in G is the adjoint of the one which acts on ϕ in (1.3) and differs from it when $H^a \neq 0$ [6].

If we choose to study the behaviour of the field at points to the future of a Cauchy hypersurface $\mathcal{S}$ on which data are assigned, it is convenient to work with the advanced Green function, which satisfies $G(x, x') = 0$ when x' is to the past of x. (Of course, the whole discussion can be carried out as well for the 'time-reversed' situation.) In general, since the wave equation gives rise to causal propagation ([13]; see also [14], p 250), the advanced Green function can be written as

$$G(x, x') = \tilde{\Sigma}(x, x') + \tilde{\Delta}(x, x') \tag{1.6}$$

where $\tilde{\Sigma}$ is a distribution term with support on light-like separated pairs (x, x') and $\tilde{\Delta}$ has support on time-like related pairs, with x to the past of x' in both cases (in fact, this is a possible *definition* of causal propagation). Thus, $\tilde{\Delta}$ is of the form

$$\tilde{\Delta}(x, x') =: \Delta(x, x')\theta_+(-\Gamma(x, x')) \tag{1.7}$$

where $\Gamma(x, x')$ is the square of the proper distance calculated along the unique geodesic connecting x and x', and θ_+, with a common abuse of notation, represents an 'advanced step function', which is zero when x lies in the future of x' [6, 12].

We will allow the Cauchy hypersurface $\mathcal{S}$ to be piecewise smooth, like, for example, the future null cone of a point, which is not differentiable at the vertex. The field ϕ at each $x \in D^+(\mathcal{S}) \equiv J^+(\mathcal{S})$ can then be expressed as

$$\phi(x) = -\int_{\mathcal{S}} \mathrm{d}S_a(x') \left[g^{ab}(x')G(x', x) \overleftrightarrow{\nabla}'_b \phi(x') + H^a(x')G(x', x)\phi(x') \right] \tag{1.8}$$

where $\mathrm{d}S_a(x')$ is the oriented volume element on the hypersurface $\mathcal{S}$ at x'; this expression is valid even if $\mathcal{S}$ is a (partially) null hypersurface, provided that $\mathrm{d}S_a$ is appropriately defined (see section 5).

We start in section 2 with a general discussion of the two-dimensional case. In section 3, we give some definitions of properties of wave propagation related to the absence of tails, whose meanings and mutual relationships are investigated in sections 4–7. Section 8 contains some concluding remarks, as well as speculations about possible lines of future research on the topic. As we have already been doing, we will use latin indices a, b, ... as abstract indices in spacetime [14], which just indicate the tensorial nature of an object without requiring a set of coordinates, while greek indices μ, ν, ... will be used for equations valid in some chart. The notations $D^\pm(\mathcal{A})$, $J^\pm(\mathcal{A})$ and $I^\pm(\mathcal{A})$, where $\mathcal{A}$ is some subset of $\mathcal{M}$, stand, respectively, for its future/past domain of dependence, causal future/past, and chronological future/past [14], all defined in terms of the causal relations induced by g_{ab}. Minkowskian coordinates, in which the coefficients of the metric have (at least at a point) the form $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, will be denoted by (t, x^i), with $i = 1, \ldots, m-1$.

2. The two-dimensional wave equation

We begin with a general discussion of the wave equation (1.3) in a two-dimensional spacetime. The motivation for devoting an entire section to such a specific subject is twofold. First, it will provide us with a concrete example with which to illustrate some general ideas. Second, and more important, much of the paper will turn out to deal exclusively with this case; it is thus convenient to establish a few preliminary points to which we can refer to later.

Any two-dimensional spacetime is conformally flat and its metric can, therefore, be locally written as $g_{ab} = \Omega^2\eta_{ab}$, where Ω is a non-vanishing function and η_{ab} is the Minkowski metric. In a general m-dimensional conformally flat spacetime we have

$$\nabla_a X^a = \partial_a X^a + \partial_a \ln|\Omega|^m X^a = |\Omega|^{-m}\partial_a(|\Omega|^m X^a) \tag{2.1}$$

for an arbitrary vector field X^a. In the particular case $m = 2$, and for $X^a = g^{ab}\nabla_b f = \Omega^{-2}\eta^{ab}\partial_b f$, equation (2.1) allows us to write

$$g^{ab}\nabla_a\nabla_b f = \Omega^{-2}\eta^{ab}\partial_a\partial_b f \tag{2.2}$$

for any function f. Therefore, in $1+1$ dimensions

$$g^{ab}\nabla_a\nabla_b f + H^a\nabla_a f + Kf = \Omega^{-2}(\eta^{ab}\partial_a\partial_b f + \bar{H}^a\partial_a f + \bar{K}f) \tag{2.3}$$

where $\bar{H}^a := \Omega^2 H^a$ and $\bar{K} := \Omega^2 K$. Since $\Omega \neq 0$ everywhere on $\mathcal{M}$, we have that ϕ satisfies the wave equation (1.3) in $(\mathcal{M}, g)$ iff it satisfies

$$\eta^{ab}\partial_a\partial_b\phi + \bar{H}^a\partial_a\phi + \bar{K}\phi = 0 \tag{2.4}$$

in the flat spacetime $(\mathcal{M}, \eta)$. We can therefore restrict ourselves to studying (2.4) without loss of generality.

It is convenient to introduce locally on $\mathcal{M}$ the null coordinates

$$u := \tfrac{1}{2}(t - x) \tag{2.5}$$

$$v := \tfrac{1}{2}(t + x) \tag{2.6}$$

in which the Minkowski metric and its inverse have only one independent non-vanishing component each, $\eta_{uv} = -2$ and $\eta^{uv} = -\frac{1}{2}$, respectively, so that (2.4) becomes

$$\partial^2_{uv}\phi + U\partial_u\phi + V\partial_v\phi + W\phi = 0 \tag{2.7}$$

with $U := -\bar{H}^u$, $V := -\bar{H}^v$ and $W := -\bar{K}$. In the rest of the paper, we shall often refer to this form of the two-dimensional wave equation.

It is not difficult to check, using (2.1) again, that for an arbitrary function f in two dimensions

$$g^{ab}\nabla_a\nabla_b f - \nabla_a\left(H^a f\right) + Kf = -\Omega^{-2}\left[\partial^2_{uv}f - \partial_u(Uf) - \partial_v(Vf) + Wf\right]. \tag{2.8}$$

Since $\delta(u, v; u', v') = \Omega^{-2}\delta(u - u')\delta(v - v')$, it follows that $G(u, v; u', v')$ is a Green function for (1.3) iff it satisfies the equation

$$\partial^2_{uv}G - \partial_u(UG) - \partial_v(VG) + WG = \delta(u - u')\delta(v - v'). \tag{2.9}$$

By direct substitution, we can verify that

$$G(u, v; u', v') = \Delta(u, v; u', v')\theta(u' - u)\theta(v' - v) \tag{2.10}$$

where θ is the step function and satisfies (2.9) provided that the following conditions are fulfilled:

$$\partial^2_{uv}\Delta - \partial_u(U\Delta) - \partial_v(V\Delta) + W\Delta = 0 \tag{2.11}$$

$$\Delta(u, v'; u', v') = \exp\left(-\int_u^{u'} \mathrm{d}u''\, V(u'', v')\right) \tag{2.12}$$

$$\Delta(u', v; u', v') = \exp\left(-\int_v^{v'} \mathrm{d}v''\, U(u', v'')\right). \tag{2.13}$$

Since (2.12) and (2.13) can be regarded as data in a characteristic initial-value problem for (2.11), which has a unique solution, it follows that $\Delta(u, v; u', v')$ is completely determined by (2.11)–(2.13); therefore, (2.10) is the general form of the advanced Green function for the two-dimensional wave equation. In the particular case $U = V = K = 0$, we recover the well known result $\Delta = 1$ [5, 7].

3. Definitions

Various properties are used to characterize the absence of wave tails. First of all, we need to distinguish between the fact that a particular solution of the wave equation may not have tails and a possible intrinsically non-diffusive nature of the equation itself. We say that a set of Cauchy data with compact support C *produces no tails* iff the field ϕ obtained evolving these data vanishes in $K^+(C)$, the set of all points in the causal future of C which cannot be reached from C by a null geodesic [15]. For such data and points $x \in K^+(C)$, (1.8) becomes

$$\phi(x) = -\int_C \mathrm{d}S_a(x') \left[g^{ab}(x')\,\Delta(x', x)\,\overset{\leftrightarrow}{\nabla}{}'_b\phi(x') + H^a(x')\Delta(x', x)\phi(x')\right]\theta(-\Gamma(x', x)) \tag{3.1}$$

since the $\tilde{\Sigma}$ term and the derivative of the θ-function in $G(x', x)$ do not contribute to the field at x (we retain the factor $\theta(-\Gamma)$ in order to also cover cases in which $x \in K^+(C)$ is not in the causal future of some $x' \in C$, as may happen, for example, when C is not simply connected). Therefore, the data on C produce no tails iff the integral on the right-hand side of (3.1) vanishes.

A related notion, expressed directly in terms of the field rather than of the Cauchy data, generalizes the concept of distortionless propagation without necessarily referring to a differential equation [5]. We say that a wave *propagates without distortion* if it can be written as $\phi(x) = R(u(x))$, where the *phase* u is a function on $\mathcal{M}$ and the *wave form* R is a function of one variable. Clearly, if R has compact support $[u_1, u_2]$, the wave propagates permanently sandwiched between those two values of u. We have here no metric with which to verify whether this wave has tails in the sense defined above, but it is nevertheless obvious that 'there are no tails with respect to u'. A slightly more general situation is that of *relatively undistorted* or *simple progressing* waves, those of the form $\phi(x) = f(x)R(u(x))$, where the *amplitude* f is a function on $\mathcal{M}$ [5, 6, 16–19]. An example in Minkowski spacetime is the spherical wave $\phi(t, \boldsymbol{x}) = \exp \mathrm{i}(k|\boldsymbol{x}| - \omega t)/|\boldsymbol{x}|$, which progresses with speed ω/k in the radial direction and is undistorted except for the decrease in the amplitude $1/|\boldsymbol{x}|$†. A finite

† Progressing waves become interesting for suitable choices of the phase; the most common ones in Minkowski spacetime have $u(t, \boldsymbol{x}) = t - F(\boldsymbol{x})$ for some wave front F, where F is a function of the spatial coordinates [16].

sum of simple progressing waves will also be called (relatively) undistorted, the simplest example being the general solution $\phi(t,x) = R(t-x) + S(t+x)$ of the two-dimensional equation (2.7) with $U = V = W = 0$. A useful further generalization is that of *progressing waves of order* N; those that can be written as

$$\phi(x) = \sum_{i=0}^{N} f_i(x) R^{(i)}(u(x)) \tag{3.2}$$

where the superscript (i) denotes the ith derivative. For these waves, it is still true that a compact support for R implies a 'sandwich propagation' for ϕ.

Turning now to intrinsic properties of the wave equation, corresponding to the absence of tails in the general solution, the most frequently used intrinsic property by far is Huygens' principle (HP)†.

HP. A wave equation is said to satisfy Huygens' principle (HP) iff the field at a point x depends only on the data at the intersection of the past light cone through x with the Cauchy hypersurface [4–6], in the sense that any two sets of data which coincide there must give the same field at x.

It is obvious from (1.8) that HP is equivalent to the requirement that the advanced Green function $G(x, x')$ has support only on pairs of points such that x lies on the past light cone of x'. The latter statement is often also referred to as HP. An obvious consequence of (2.10) is that HP is always violated in a two-dimensional spacetime.

The concepts discussed at the beginning of this section give rise, however, to other definitions that can be found in the literature. Using the notion of data that produce no tails, we define:

TF. A wave equation is said to be *tail-free* (TF) iff each set of Cauchy data with compact support produces no tails [15].

It is almost trivial to see that TF and HP are equivalent; nevertheless, for the sake of clarity, we shall present the explicit proof in the next section.

It is sometimes interesting to consider data assigned on a characteristic (i.e. null [6]) hypersurface, and to give, therefore, a modified version of TF. This property was originally formulated in terms of solutions of a wave equation [20], but we find it more meaningful to state it as applied to the equation itself [15, 21]:

CPP. A wave equation is said to satisfy the *characteristic propagation property* (CPP) iff each set of null data of compact support, which vanish at any singular points of the support, produce no tails.

† The improper use of the term 'principle' to denote what is actually only a property is commonly adopted for historical reasons, with the probably unique exception of Hadamard, who refers to it as 'Huygens' minor premise' [4]. In reality, Huygens made use of TF below rather than HP, and merely in an *approximate* version by postulating that *almost* all the waves emitted by a point-like source are concentrated on the wavefront (see [1], e.g., pp 18 and 22). It is interesting to notice that this hypothesis does not correspond to assuming that tails are absent, but only that they are small!

Why restrict, in this definition, the admissible data to those that vanish at singular points? For a two-dimensional characteristic initial-value problem, we can consider, without loss of generality, the initial data $\phi(u,0) =: \varphi(u)$ and $\phi(0,v) =: \psi(v)$ to be assigned on the union of the two 'half-axes' $u \geqslant 0$ and $v \geqslant 0$. Consider the simplest example of (2.7), with $U = V = W = 0$. Its general solution is of the form $\phi(u,v) = R(u) + S(v)$, which in terms of the data becomes $\phi(u,v) = \varphi(u) + \psi(v) - \phi(0,0)$. This means that, if $\phi(0,0) \neq 0$, no data with compact supports $[u_1, u_2]$ and $[v_1, v_2]$ can lead to 'sandwich propagation', i.e. one for which the support of the full solution is contained in the union of the strips $u_1 \leqslant u \leqslant u_2$ and $v_1 \leqslant v \leqslant v_2$. There is, of course, nothing bad in this, since we know already that HP is violated in two dimensions. However, data $\varphi(u)$ and $\psi(v)$ such that $\varphi(0) = \psi(0) = \phi(0,0) = 0$ do lead to sandwich propagation. Here, the point $(0,0)$ is an example of a singular point of S, and the reason we excluded data which do not vanish at such points in the definition of CPP is that this allows us to give a reasonable no-tails characterization that may hold even in cases in which HP does not. Actually, we shall see in section 5 that CPP, although in principle weaker than HP and TF, in practice, always implies them, except in two spacetime dimensions. We wish to stress that the restriction of data we are considering is actually a very small one, concerning a subset of C of measure zero which contains points that are already pathological. Furthermore, such a restriction corresponds precisely to what is commonly performed when assigning data on the asymptotic past (probably the only characteristic initial-value problem with good physical motivations), where one allows $\phi \neq 0$ at past null infinity $\mathcal{I}^-$ but requires $\phi = 0$ at past time-like infinity i^- [14].

In addition, the notion of progressing waves motivates the following definition for two-dimensional equations:

PW_N. A wave equation in two spacetime dimensions is said to satisfy the *progressing-wave propagation property of order* N (PW_N) iff its general solution can be written as a sum of two progressing waves

$$\phi(x) = \sum_{i=0}^{N} f_i(x) R^{(i)}(u(x)) + \sum_{i=0}^{N} g_i(x) S^{(i)}(v(x)) \tag{3.3}$$

where the amplitudes f_i and g_i are fixed functions on $\mathcal{M}$ depending on the wave equation (and at least one of f_N and g_N is not identically zero), while the wave forms R and S are arbitrary sufficiently differentiable functions of one variable (this forces the coordinates u and v to be null) [17, 19, 22].

In section 6, we shall show that, in two spacetime dimensions, CPP is equivalent to PW_0; as far as the PW_N, with $N > 0$, are concerned, their resemblance to a no-tails property is only apparent. Furthermore, wave equations in more than two dimensions may well have progressing-wave solutions for appropriate choices of the wave front (see, e.g., [16]), but it seems excessive to ask that their *general* solution be expressible as a finite sum of progressing waves. These issues, which diminish the appeal of PW_N as regards to the study of tails, will be discussed in the concluding section.

Another property that can be satisfied by two-dimensional wave equations is their solvability by the method of Kundt and Newman [20]. To apply this method, we start by writing (2.7) in the two equivalent normal forms

$$\partial_v(j_0 \partial_u \phi_0) - j_1 \phi_0 = 0 \tag{3.4}$$

$$\partial_u(l_0 \partial_v \psi_0) - l_{-1} \psi_0 = 0 \tag{3.5}$$

where ϕ_0 and ψ_0 are related to ϕ by factor transformations $\phi_0 = \phi \exp \sigma$ and $\psi_0 = \phi \exp \tau$, with

$$\sigma(u, v) := \int^u \mathrm{d}u' \, V(u', v) \tag{3.6}$$

$$\tau(u, v) := \int^v \mathrm{d}v' \, U(u, v') \tag{3.7}$$

while the coefficients are related to those in (2.7) by

$$j_0 = l_0^{-1} = \exp(\tau - \sigma) \tag{3.8}$$

$$j_1 = (\partial_v V + UV - W) j_0 \tag{3.9}$$

and

$$l_{-1} = (\partial_u U + UV - W) l_0. \tag{3.10}$$

If we inductively define j_k and ϕ_k by

$$j_{k+1}/j_k = j_k/j_{k-1} - \partial^2_{uv} \ln |j_k| \tag{3.11}$$

$$j_{k+1}\phi_{k+1} = j_k \partial_u \phi_k \tag{3.12}$$

assuming of course $j_k \neq 0$ for all $k \in \mathbb{Z}$, we obtain from (3.4) a countable set of equations

$$\partial_v (j_k \partial_u \phi_k) - j_{k+1}\phi_k = 0 \qquad k \in \mathbb{Z}. \tag{3.13}$$

Similarly, if we inductively define l_k and ψ_k by

$$l_{k-1}/l_k = l_k/l_{k+1} - \partial^2_{vu} \ln |l_k| \tag{3.14}$$

$$l_{k-1}\psi_{k-1} = l_k \, \partial_v \psi_k \tag{3.15}$$

assuming now that $l_k \neq 0$ for all $k \in \mathbb{Z}$, we obtain from (3.5) a second countable set

$$\partial_u (l_k \partial_v \psi_k) - l_{k-1}\psi_k = 0 \qquad k \in \mathbb{Z}. \tag{3.16}$$

We shall refer to equations (3.13) and (3.16) as being in the kth v- and u-normal form, respectively. It is not hard to check that for all $k \in \mathbb{Z}$ the kth v-normal-form equation, corresponding to the coefficients j_k and j_{k+1}, and the kth u-normal-form equation, corresponding to the coefficients l_k and l_{k-1}, are equivalent under the transformation

$$j_k \, l_k = 1 \tag{3.17}$$

$$\phi_k = l_k \, \psi_k \tag{3.18}$$

for $k \in \mathbb{Z}$. It is also easy to see that the equations within the set (3.13) (respectively, (3.16)) are equivalent in the sense that a solution of any one of them generates a solution of every other one of them through (3.11)–(3.13), (3.17), and (3.18) (respectively, (3.14)–(3.18)) and we thus obtain two equivalence classes of v- and u-normal-form equations labelled by the index k ranging over both negative and positive integers.

Given a wave equation in the 0th v-normal form (3.4), we say that its substitution sequence $\{j_k\}$ is *double terminating in N steps* [20] when $j_{k_1+1} = 0$ and $l_{k_2-1} = 0$ (but $j_{k_1} \neq 0$ and $l_{k_2} \neq 0$) for some $k_1 \geqslant 0$, $k_2 \leqslant 0$ and $N = \max\{k_1, -k_2\}$. However, it was shown in [20], using (3.11)–(3.18), that in this case the general solution of (3.4) is $\phi_0 = \phi_A + l_0\phi_R$, where

$$\phi_A := \frac{1}{j_1}\partial_v\left(\frac{j_1}{j_2}\,\partial_v\left(\frac{j_2}{j_3}\,\partial_v\left(\cdots\frac{j_{k_1-1}}{j_{k_1}}\partial_v\left(j_{k_1}\,S(v)\right)\cdots\right)\right)\right) \tag{3.19}$$

$$\phi_R := \frac{1}{l_{-1}}\,\partial_u\left(\frac{l_{-1}}{l_{-2}}\,\partial_u\left(\frac{l_{-2}}{l_{-3}}\,\partial_u\left(\cdots\frac{l_{k_2+1}}{l_{k_2}}\partial_u\left(l_{k_2}\,R(u)\right)\cdots\right)\right)\right) \tag{3.20}$$

with $S(v)$ and $R(u)$ arbitrary functions of one variable. It is obvious from (3.19) and (3.20) that such a ϕ is a progressing wave of order N. This relationship motivates the following definition.

KN_N. A wave equation in two spacetime dimensions is said to be *solvable by the Kundt–Newman method in N steps* (KN_N) iff its substitution sequence is double terminating in N steps.

As we have just seen, all KN_N wave equations are PW_N. We shall see in section 7 that the converse is also true, so that KN_N and PW_N are equivalent properties.

4. Equivalence between HP and TF

We now begin studying the relationships between the various properties of wave equations listed in the previous section. The first three are related in a simple way in any number of dimensions; as we will now show, TF and HP are always trivially equivalent, which justifies the fact that they are often identified, and they are almost always equivalent to CPP (see next section). These results generalize the arbitrary wave equations of [15] to those of type (1.3).

In order to show that TF $\Rightarrow$ HP, let us consider any point $x \in D^+(\mathcal{S})$ and assign data with support $\mathcal{C} \subset I^-(x) \cap \mathcal{S}$. Since TF holds by hypothesis, and $x \in K^+(\mathcal{C})$, it follows that $\phi(x) = 0$ for each set of data prescribed on such a $\mathcal{C}$. By causality, $\phi(x)$ cannot be influenced by data given outside $I^-(x) \cap \mathcal{S}$ and, hence, it can depend only on their value on $E^-(x) \cap \mathcal{S}$, where $E^-(x)$ stands for the set of points in $J^-(x)$ that are null related to x. This is precisely the content of HP.

Let us now prove that HP $\Rightarrow$ TF. First of all, let us notice that, by choosing data that vanish everywhere on $\mathcal{S}$, we get $\phi(x) = 0$ everywhere in $\mathcal{M}$ by (1.3). If we now assign data on $\mathcal{S}$ with compact support $\mathcal{C}$, HP implies that the value of $\phi(x)$, for any $x \in K^+(\mathcal{C})$, must be independent of the data; in particular, it must have the same value than in the case of vanishing data, i.e. it must be equal to zero, by the remark above. This completes the proof that HP $\Leftrightarrow$ TF.

It is interesting to notice that the structure of this proof allows one to extend it to more general wave equations than (1.3). Actually, in the case of the latter, a simpler proof can be given that makes use of the equivalence between HP and the property $\tilde{\Delta}(x', x) \equiv 0$. In fact, it follows immediately from (3.1) that, if HP is satisfied, then $\phi(x) = 0$ for all $x \in K^+(\mathcal{C})$, i.e. the wave equation is TF. The converse is also true, as we can see by choosing $\mathcal{S}$ to be space-like at a point $\bar{x}$, and data of support only at $\bar{x}$, such that, denoting the unit vector normal to $\mathcal{S}$ by n^a

$$\phi|_{\mathcal{S}}(x) \equiv 0 \tag{4.1}$$

$$(n^a \nabla_a \phi)|_{\mathcal{S}}(x) = \delta_{\mathcal{S}}(x, \bar{x}) \tag{4.2}$$

where $\delta_{\mathcal{S}}$ is the $(m-1)$-dimensional delta function on $\mathcal{S}$. Then, (3.1) reduces to

$$\phi(x) = -\Delta(\bar{x}, x) \tag{4.3}$$

and if $\phi(x) = 0$ for all $x \in K^+(C) = I^+(\bar{x})$, we must have $\Delta(\bar{x}, x) = 0$ for all such pairs of points, i.e. $\tilde{\Delta}(x', x) \equiv 0$. However, this proof relies heavily on the existence of the integral representation (1.8) and appears much more difficult to generalize to the case of more complicated wave equations than the 'geometrical' one given above.

5. Relationship between HP and CPP

It is obvious from the definitions given in section 3 that TF, and thus HP, imply CPP. To check whether the converse holds, we need to consider again the integral expression (3.1) for $x \in K^+(C)$, where C is now part of a piecewise null hypersurface $\mathcal{S}$ locally defined by the condition $w =$ constant, for some $w : \mathcal{M} \to \mathbb{R}$ with $g^{ab}\nabla_a w \nabla_b w = 0$ at all points where $\mathcal{S}$ is differentiable (this condition may fail only in a subset of $\mathcal{S}$ of measure zero). One usually writes $\mathrm{d}S_a = \mathrm{d}S\, n_a$, where $\mathrm{d}S$ and n_a are often thought of as volume element and unit normal to the surface $\mathcal{S}$, respectively, which clearly does not make sense if $\mathcal{S}$ is null. However, the definitions of these objects can be extended to that case as well. Let us introduce local coordinates $\{\xi^i : i = 1, \ldots, m-1\}$ on $\mathcal{S}$ and use w as a coordinate transverse to $\mathcal{S}$, so that (w, ξ^i) are coordinates on $\mathcal{M}$ adapted to $\mathcal{S}$. Then we can always write $\mathrm{d}S_\mu = \mathrm{d}^{m-1}\xi \sqrt{-g}\,\delta_\mu{}^w$, with the understanding that g must be calculated in these coordinates, which in the null case can be split into $\mathrm{d}S = \mathrm{d}^{m-1}\xi \sqrt{-g}$ and $n_\mu = \delta_\mu{}^w$; the latter are the components of the form $n_a = \nabla_a w$. With these conventions, we can write the Gauss theorem for an arbitrary vector field Y^a as

$$\int_{\mathcal{N}} \mathrm{d}V\, \nabla_a Y^a = \oint_{\partial\mathcal{N}} \mathrm{d}S_a\, Y^a \tag{5.1}$$

where $\mathcal{N}$ is a region in $\mathcal{M}$ and $\mathrm{d}V = \mathrm{d}^m x \sqrt{-g}$ is the spacetime volume element, even when (part of) $\partial\mathcal{N}$ is null. This follows from the fact that the left-hand side of (5.1) can be broken into a sum of integrals of the form

$$\int_{\mathcal{D}} \mathrm{d}^{m-1}\xi\, \mathrm{d}w\, \partial_\mu(\sqrt{-g}Y^\mu) \tag{5.2}$$

where $\mathcal{D}$ is a domain in $\mathbb{R}^m$, one of whose 'faces' (which we denote by $\mathcal{F}$) is defined by $w = 0$; but (5.2) can be transformed, using the Gauss theorem in $\mathbb{R}^m$, into a sum of terms corresponding to the various faces of $\mathcal{D}$, of which only

$$\int_{\mathcal{F}} \mathrm{d}^{m-1}\xi \sqrt{-g}\,\delta_\mu{}^w Y^\mu \tag{5.3}$$

contributes, in the end, to the total expression. Therefore, (5.1) (and hence (1.8) and (3.1)) is completely justified, independently of the causal character of $\partial\mathcal{N}$.

We can rewrite (3.1) in the coordinates (w, ξ^i) as

$$\phi(x) = -\int_C \mathrm{d}S' \left\{\nabla'_\mu(n^\mu \Delta\phi) - \phi\left[(\nabla'_\mu n^\mu)\Delta + n^\mu(2\nabla'_\mu\Delta - H_\mu\Delta)\right]\right\}\theta(-\Gamma). \tag{5.4}$$

(Here, and in the following integrals, all functions in the integrand depend on x' and all two-point functions on the ordered pair (x', x).) Since we have $n^\mu = g^{\mu w}$, which implies $n^w = g^{ww} = g^{\mu\nu}\partial_\mu w \partial_\nu w = 0$, the first term on the right-hand side of (5.4) contains only derivatives performed along S and can be transformed as

$$-\int_C \mathrm{d}S'\, \nabla'_\mu(n^\mu\, \Delta\, \phi) = -\int_{\mathcal{F}} \mathrm{d}^{m-1}\xi'\, \partial'_i(\sqrt{-g}n^i\, \Delta\phi) = -\oint_{\partial C} \mathrm{d}\sigma'_a\, n^a\, \Delta\phi \tag{5.5}$$

where $\mathcal{F}$ stands again for the region of $\mathbb{R}^m$ defined by $w = 0$, $\mathrm{d}\sigma_a$ is the oriented volume element on ∂C and the Gauss theorem in $\mathbb{R}^{m-1}$ has been applied in the last step (possible discontinuities of ϕ do not prevent us from using the theorem if they are correctly interpreted in the sense of distributions). We can, therefore, deduce that this term does not contribute to the expression, because $\phi(x') = 0$ outside C and we can think of the integration as being performed over a region larger than C, on whose boundary $\phi(x') = 0$. Therefore, CPP is satisfied iff

$$\int_C \mathrm{d}S'\, \phi[(\nabla'_\mu n^\mu)\Delta + n^\mu(2\nabla'_\mu\Delta - H_\mu\Delta)]\theta(-\Gamma) = 0 \tag{5.6}$$

for all data on an arbitrary compact null C. In particular, we can choose as data

$$\phi|_S(x) = \delta_S(x, \bar{x}) \tag{5.7}$$

for any non-singular point $\bar{x} \in C$. Since $\bar{x}$ is arbitrary, this implies that either $\Delta(x', x) = 0$ and we get HP directly, or

$$\nabla_a n^a(x) + n^a(x)[2\nabla_a \ln|\Delta(x, x')| - H_a(x)] = 0 \tag{5.8}$$

for all pairs of points with $x' \in I^+(x)$, $x \in C$ non-singular and all null vector fields n^a which are gradients of a function.

We will now study the consequences of (5.8) by writing it in a suitable coordinate system. In terms of w, (5.8) is of the form

$$g^{ab}\nabla_a\nabla_b w + X^a\nabla_a w = 0 \tag{5.9}$$

where $X^a(x, x') := 2\nabla^a \ln|\Delta(x, x')| - H^a(x)$. Choose any point $x_0 \in C$, and a system of Riemann normal coordinates based at x_0, for which $g_{\mu\nu}(x) = \eta_{\mu\nu} + \mathcal{O}(2)$, where we have introduced the notation $\mathcal{O}(n)$ for a quantity which is of order n in the coordinate separation of x from x_0; then (5.9) becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu w + X^\mu\partial_\mu w = \mathcal{O}(1). \tag{5.10}$$

Let us now consider the function

$$w = t + \sqrt{x^i x_i} + c_{\mu\nu}x^\mu x^\nu + \mathcal{O}(3) \tag{5.11}$$

where the $c_{\mu\nu}$ are appropriate coefficients such that $g^{\mu\nu}\partial_\mu w \partial_\nu w = 0$. Substituting into (5.10), we get that, for $x^i x_i \neq 0$,

$$\frac{m-2}{\sqrt{x^i x_i}} + 2\eta^{\mu\nu} c_{\mu\nu} + X^t(x, x') + \frac{\eta_{ij} X^i(x, x') x^j}{\sqrt{x_k x^k}} = \mathcal{O}(1). \tag{5.12}$$

The first term on the left-hand side of (5.12) is $\mathcal{O}(-1)$, whereas the others are all $\mathcal{O}(0)$; therefore, (5.12) implies $m = 2$ and

$$X^t(x_0, x') + X^x(x_0, x') \lim_{x \to 0^\pm} \frac{x}{|x|} = -2\eta^{\mu\nu} c_{\mu\nu} \tag{5.13}$$

where the $\pm$ in the limit corresponds to the arbitrariness in the direction of approach of x^μ to the origin and, with some abuse of notation, we have denoted the spatial coordinate of the two-dimensional Minkowski spacetime by x. This means that

$$X^t(x_0, x') \pm X^x(x_0, x') = -2\eta^{\mu\nu} c_{\mu\nu} \tag{5.14}$$

must hold simultaneously for both signs, i.e. $X^x(x_0, x') = 0$. Since this is true in any system of local Minkowskian coordinates, and taking into account the arbitrariness of x_0, we deduce that $X^a(x, x') = 0$ as a field.

In conclusion, we have found that CPP implies either HP or $m = 2$ and

$$H_a(x) = 2\nabla_a \ln |\Delta(x, x')|. \tag{5.15}$$

In other words, CPP is always equivalent, for $m > 2$, to HP; in two spacetime dimensions $\Delta \neq 0$ and HP is always violated, whereas CPP can be satisfied. Therefore, CPP can be regarded as a non-trivial generalization of HP, although it becomes interesting by itself only for $m = 2$. More explicitly, the definitions of CPP and HP differed in that, for the former, the support of the data had to be null rather than achronal and the data had to vanish at non-smooth points of $\mathcal{C}$. We now know that this is of no consequence other than in some two-dimensional cases, but it is perhaps surprising that such a small restriction in the data can have the effect of enlarging, in a non-trivial way, the class of equations to which the definition applies.

We have derived (5.15) above as a necessary condition for the validity of CPP in two dimensions, but the following simple argument shows that it is also sufficient. Substituting (5.15) into (5.9), the latter reduces to $g^{ab}\nabla_a\nabla_b w = 0$ and CPP holds if this equation is satisfied by any function w that generates null hypersurfaces. For $m = 2$, this is true because w is null iff it depends on only one of the coordinates u and v, defined by (2.5) and (2.6), i.e. iff either $w = w_1(u)$ or $w = w_2(v)$. Using identity (2.2) for $f = w$, we have $g^{ab}\nabla_a\nabla_b w = 0$.

Let us now see what restrictions condition (5.15) places on the wave equation. First of all, since the Riemannian connection is torsion-free, we have $\nabla_{[a}\nabla_{b]} \ln|\Delta| = 0$ and (5.15) implies that

$$\nabla_{[a} H_{b]} = 0. \tag{5.16}$$

This is an integrability condition that allows us to obtain Δ by straightforward integration of (5.15). In fact, (5.16) implies that $H_a = 2\nabla_a \Lambda$, for some Λ; substituting back into (5.15), we have

$$|\Delta(x, x')| = \frac{\exp \Lambda(x)}{\exp \Lambda(x')} \tag{5.17}$$

since $\Delta(x', x') = 1$ by (2.9) and (2.10). Furthermore, it is not difficult to verify, from the definition of the Green function, that $\Delta(x, x')$ satisfies the 'adjoint wave equation' in x (see also (2.8) and (2.11))

$$g^{ab}\nabla_a\nabla_b\Delta - \nabla_a(H^a\Delta) + K\Delta = 0. \tag{5.18}$$

Substituting (5.15) twice into (5.18), we get

$$2\nabla_a H^a + g_{ab}H^a H^b - 4K = 0 \tag{5.19}$$

and hence

$$g^{ab}\nabla_a\nabla_b\Lambda + g^{ab}\nabla_a\Lambda\nabla_b\Lambda - K = 0. \tag{5.20}$$

Using (5.17), we can replace (5.20) by an equation for Δ which is simpler than (5.18)

$$g^{ab}\nabla_a\nabla_b\Delta - K\Delta = 0 \tag{5.21}$$

which can also be directly obtained by using (5.19) in (5.18). Equations (5.16) and (5.20) (or (5.19)) are also sufficient conditions for CPP, as one can easily see by considering a new field $\psi := \phi \exp\Lambda$. From (1.3) and (5.20), we find that ψ obeys the equation $g^{ab}\nabla_a\nabla_b\psi = 0$, which is CPP because $m = 2$. Since ψ and ϕ differ only by the non-vanishing factor $\exp\Lambda$, it follows that ϕ satisfies CPP as well.

In the coordinates (u, v), we have $H^u = -\Omega^{-2}U$, $H^v = -\Omega^{-2}V$, $K = -\Omega^{-2}W$, so that (5.16) and (5.19) are equivalent to

$$\partial_u U = \partial_v V \tag{5.22}$$

and

$$\partial_u U + UV - W = 0 \tag{5.23}$$

respectively. Similarly, we have $U = \partial_v\Lambda$ and $V = \partial_u\Lambda$; (5.20) and (5.21) become

$$\partial^2_{uv}\Lambda + \partial_u\Lambda\partial_v\Lambda - W = 0 \tag{5.24}$$

and

$$\partial^2_{uv}\Delta - W\Delta = 0. \tag{5.25}$$

These conditions for CPP represent strong constraints on the wave equation; for example, when $U = V = 0$, (5.23) and (5.24) are satisfied only if $W = 0$ [15]. Notice that when (5.16), or equivalently (5.22), is satisfied, the functions σ and τ defined in (3.6) and (3.7) both reduce to Λ; upon substitution into (3.8)–(3.10), equations (5.22) and (5.23) then imply that $j_0 = l_0 = 1$ and $j_1 = l_{-1} = 0$.

6. Equivalence between CPP and PW_0 in two spacetime dimensions

In this section we will show that, in two spacetime dimensions, CPP and PW_0 are equivalent properties; the proof is very simple if we rely on some results of the previous section. In fact, since the two-dimensional wave equation (2.7) satisfies CPP iff ϕ can be written as $\phi = \exp(-\Lambda)\psi$, with $\partial^2_{uv}\psi = 0$, it follows that (2.7) is CPP iff its general solution has the form

$$\phi(u, v) = \mathrm{e}^{-\Lambda(u,v)}(r(u) + s(v)) \tag{6.1}$$

where r and s are arbitrary functions. By comparing (6.1) with the $N = 0$ case of (3.3)

$$\phi(u, v) = f_0(u, v)R(u) + g_0(u, v)S(v) \tag{6.2}$$

we conclude immediately that $\mathrm{CPP} \Rightarrow \mathrm{PW}_0$. In order to prove the converse, i.e. that $\mathrm{PW}_0 \Rightarrow \mathrm{CPP}$, substitute expression (6.2) into (2.7); invoking the arbitrariness of R and S, we get

$$\partial^2_{uv} f_0 + U\partial_u f_0 + V\partial_v f_0 + W f_0 = 0 \tag{6.3}$$

$$\partial^2_{uv} g_0 + U\partial_u g_0 + V\partial_v g_0 + W g_0 = 0 \tag{6.4}$$

$$\partial_v f_0 + U f_0 = 0 \tag{6.5}$$

$$\partial_u g_0 + V g_0 = 0. \tag{6.6}$$

After some manipulations, these equations lead to (5.22) and (5.23), which are sufficient conditions for the validity of CPP, as we saw in section 5. We can therefore conclude that, in a two-dimensional spacetime, CPP and PW_0 are equivalent.

An immediate corollary of this result is that not every expression of the type (6.2) is a general solution of a two-dimensional wave equation. This is evident by a comparison between (6.1) and (6.2), which shows that the amplitudes f_0 and g_0 must be related by $f_0(u, v)/g_0(u, v) = \alpha(u)/\beta(v)$, for some functions α and β. The same conclusion can be reached in a more explicit way by integration of (6.5) and (6.6) to obtain

$$f_0(u, v) = f_0(u, 0) \exp\left(-\int_0^v \mathrm{d}v'\, U(u, v')\right) = \alpha(u) \exp(-\Lambda(u, v)) \tag{6.7}$$

$$g_0(u, v) = g_0(0, v) \exp\left(-\int_0^u \mathrm{d}u'\, V(u', v)\right) = \beta(v) \exp(-\Lambda(u, v)) \tag{6.8}$$

with $\alpha(u) := f_0(u, 0) \exp\Lambda(u, 0)$ and $\beta(v) := g_0(0, v) \exp\Lambda(0, v)$. By a generalization of this argument, one could show that not every expression of the type (3.3), with arbitrary R and S, is a general solution of a two-dimensional wave equation.

It is possible to prove that $\mathrm{CPP} \Leftrightarrow \mathrm{PW}_0$ without necessarily using the results of the previous section. This is obvious for what concerns the implication $\mathrm{PW}_0 \Rightarrow \mathrm{CPP}$, since PW_0 leads us to the same equations (5.22) and (5.23) of section 5 and one can proceed to show that they are sufficient conditions for CPP in exactly the same way as before (see section 5). The proof that $\mathrm{CPP} \Rightarrow \mathrm{PW}_0$ is less compact, but we nevertheless present it because of its intrinsic interest.

Consider the null data of compact support given by $\varphi(u) = \delta(u - u_0)$, with $u_0 > 0$ and $\psi(v) \equiv 0$; then CPP implies that $\phi(u, v)$ must be concentrated on the line $u = u_0$. This means that, for any fixed v, $\phi(u, v)$ must be a distribution in u that, applied to a test function $F(u, v)$, produces a number depending only on the behaviour of F at the point (u_0, v), i.e. on $F^{(i)}(u_0, v)$ with $i \geqslant 0$. Since distributions are linear functionals, the combination of the various $F^{(i)}(u_0, v)$ must be linear, so that we can write

$$\phi(u, v) = \sum_{i=0}^{+\infty} f_i(u_0, v)\delta^{(i)}(u - u_0) \tag{6.9}$$

for some functions f_i. The sum on the right-hand side of (6.9) actually consists only of a finite number of terms; this follows from a rigorous result of distribution theory [23], but can be heuristically justified as follows. Since the action of ϕ as a functional must be defined on any test function F, the value of the series

$$\sum_{i=0}^{+\infty} f_i(u_0, v)F^{(i)}(u_0, v) \tag{6.10}$$

must be finite for each F. Suppose that the sum in (6.9) and (6.10) is infinite, i.e. there exist infinitely many $f_i \neq 0$. Then, taking F such that $F^{(i)}(u_0, v) = f_i(u_0, v)^{-1}$ when $f_i \neq 0$, and arbitrary otherwise, the expression (6.10) becomes infinite, which contradicts the hypothesis of regularity of ϕ. Hence, we must have

$$\phi(u, v) = \sum_{i=0}^{N_u} f_i(u_0, v)\delta^{(i)}(u - u_0) \tag{6.11}$$

with N_u finite. Similarly, if $\varphi(u) \equiv 0$ and $\psi(v) = \delta(v - v_0)$, we have

$$\phi(u, v) = \sum_{i=0}^{N_v} g_i(u, v_0)\delta^{(i)}(v - v_0) \tag{6.12}$$

for some finite N_v and some functions g_i.

From (6.11) and (6.12), it already follows that (2.7) satisfies PW$_N$, with $N = \max\{N_u, N_v\}$, as can be seen by using just linearity and representing arbitrary data φ and ψ as superpositions of δ-like data

$$\varphi(u) = \int \mathrm{d}u_0\, \varphi(u_0)\delta(u - u_0) \tag{6.13}$$

$$\psi(v) = \int \mathrm{d}v_0\, \psi(v_0)\delta(v - v_0) \tag{6.14}$$

but without making any other use of the differential equation for ϕ. We shall now show, *however, that* (2.7) *allows us to set* $N = 0$, *i.e. to conclude that* CPP$\Rightarrow$PW$_0$. For this purpose, let us first consider again data $\varphi(u) = \delta(u - u_0)$ and $\psi(v) \equiv 0$. Substituting the solution (6.11) into (2.7), we get

$$\begin{aligned}
&\left[\partial_v f_{N_u}(u_0, v) + U(u, v) f_{N_u}(u_0, v)\right]\delta^{(N_u+1)}(u - u_0)\\
&\qquad + \sum_{i=1}^{N_u}\left[V(u, v)\partial_v f_i(u_0, v) + W(u, v) f_i(u_0, v)\right.\\
&\qquad \left. + \partial_v f_{i-1}(u_0, v) + U(u, v) f_{i-1}(u_0, v)\right]\delta^{(i)}(u - u_0)\\
&\qquad + \left[V(u_0, v)\partial_v f_0(u_0, v) + W(u_0, v) f_0(u_0, v)\right]\delta(u - u_0) = 0.
\end{aligned} \tag{6.15}$$

Smoothing (6.15) with a test function, and taking into account the arbitrariness of the latter, we obtain the following set of equations:

$$\partial_v f_{N_u}(u_0, v) + U(u_0, v) f_{N_u}(u_0, v) = 0 \tag{6.16}$$

$$\begin{aligned}\sum_{i=k}^{N_u}(-1)^i \binom{i}{k} \big[\partial_u^{i-k} V(u_0, v)\partial_v f_i(u_0, v) + \partial_u^{i-k} W(u_0, v) f_i(u_0, v) \\ + \partial_u^{i-k} U(u_0, v) f_{i-1}(u_0, v)\big] + (-1)^k \partial_v f_{k-1}(u_0, v) \\ + (-1)^{N_u+1}\binom{N_u+1}{k}\partial_u^{N_u-k+1} U(u_0, v) f_{N_u}(u_0, v) = 0 \qquad N_u \geqslant k \geqslant 1\end{aligned} \tag{6.17}$$

$$\begin{aligned}\sum_{i=1}^{N_u}(-1)^i \big[\partial_u^i V(u_0, v)\partial_v f_i(u_0, v) + \partial_u^i W(u_0, v) f_i(u_0, v) + \partial_u^i U(u_0, v) f_{i-1}(u_0, v)\big] \\ + V(u_0, v)\partial_v f_0(u_0, v) + W(u_0, v) f_0(u_0, v) \\ = 0 + (-1)^{N_u+1}\partial_u^{N_u+1} U(u_0, v) f_{N_u}(u_0, v).\end{aligned} \tag{6.18}$$

From (6.16) and the fact that $f_{N_u}(u_0, 0) = 0$, which follows from using our data in (6.11), we obtain that $f_{N_u}(u_0, v) = 0$ for all v. Using this in (6.17) and repeating the procedure, we obtain $f_i(u_0, v) = 0$, $\forall i \geqslant 1$, so that only $f_0(u_0, v)$ can be non-zero. An analogous argument, using data $\varphi(u) \equiv 0$, $\psi(v) = \delta(v - v_0)$, shows that the only non-vanishing coefficient in (6.12) can be $g_0(u, v_0)$. Therefore, the general solution, obtained from (6.11) and (6.12) using arbitrary initial data as in (6.13) and (6.14), is

$$\phi(u, v) = f_0(u, v)\varphi(u) + g_0(u, v)\psi(v) \tag{6.19}$$

from which the validity of PW_0 follows. Furthermore, it is not difficult to see that (6.17) for $k = 1$ reduces to (6.5) in $u = u_0$, and that substituting this relation into (6.18), we obtain (5.23). By a completely symmetric procedure we can recover (6.6).

7. Equivalence between PW_N and KN_N

As we saw in section 3, all KN_N wave equations are PW_N; it has been conjectured that the converse is also true, namely that all PW_N equations can in fact be solved exactly by the KN_N method [20, 24]. To show that this is indeed the case, suppose we have a two-dimensional wave equation whose general solution is the progressing wave (3.3). By suitable transformations, we can always write the wave equation in its 0th v- or u-normal form, (3.4) or (3.5); let us concentrate on the v-normal form.

Applying the differential operator in (3.4) to the progressing wave (3.3), and imposing that the coefficients of derivatives of the arbitrary functions $R(u)$ and $S(v)$ of different orders vanish separately in the resulting expression, we have

$$\partial_v(j_0\partial_u f_0) - j_1 f_0 = 0 \tag{7.1}$$

$$\partial_v(j_0\partial_u f_i) - j_1 f_i + \partial_v(j_0 f_{i-1}) = 0 \qquad 1 \leqslant i \leqslant N \tag{7.2}$$

$$\partial_v(j_0 f_N) = 0 \tag{7.3}$$

and

$$\partial_v(j_0\partial_u g_0) - j_1 g_0 = 0 \tag{7.4}$$

$$\partial_v(j_0\partial_u g_i) - j_1 g_i + j_0\partial_u g_{i-1} = 0 \qquad 1 \leqslant i \leqslant N \tag{7.5}$$

$$\partial_u g_N = 0. \tag{7.6}$$

Let us consider the group (7.4)–(7.6). Equations (7.4) and (7.5) can be rewritten more conveniently as

$$\partial_v\left(\frac{j_0}{j_1}\partial_u g_0\right) + \partial_v \ln|j_1|\frac{j_0}{j_1}\partial_u g_0 - g_0 = 0 \tag{7.7}$$

and

$$\partial_v\left(\frac{j_0}{j_1}\partial_u g_i\right) + \partial_v \ln|j_1|\frac{j_0}{j_1}\partial_u g_i - g_i + \frac{j_0}{j_1}\partial_u g_{i-1} = 0 \qquad 1 \leqslant i \leqslant N \tag{7.8}$$

respectively. By repeated differentiation of (7.8) with respect to u, one finds that, for any $h \geqslant 1$ and $k = 0, \ldots, N-1$,

$$\partial_v \mathcal{D}_u^h g_{N-k} + \partial_v \ln|j_h| \mathcal{D}_u^h g_{N-k} - \mathcal{D}_u^{h-1} g_{N-k} + \mathcal{D}_u^h g_{N-k-1} = 0 \tag{7.9}$$

where the differential operator $\mathcal{D}_u^k$, containing k derivatives ∂_u, is defined as

$$\mathcal{D}_u^k F := \frac{j_{k-1}}{j_k}\partial_u\left(\frac{j_{k-2}}{j_{k-1}}\partial_u\left(\cdots\frac{j_1}{j_2}\partial_u\left(\frac{j_0}{j_1}\partial_u F\right)\cdots\right)\right) \tag{7.10}$$

for any sufficiently differentiable function F. We shall prove (7.9) by induction over h. Assuming that it holds for some h, and differentiating it with respect to u, we have

$$\partial^2_{uv}\mathcal{D}_u^h g_{N-k} + \partial_v \ln|j_h|\partial_u \mathcal{D}_u^h g_{N-k} - \frac{j_{h+1}}{j_h}\mathcal{D}_u^h g_{N-k} + \partial_u \mathcal{D}_u^h g_{N-k-1} = 0 \tag{7.11}$$

where we have used (3.11). Rewriting the first term as

$$\partial_v\left(\frac{j_{h+1}}{j_h}\frac{j_h}{j_{h+1}}\partial_u\mathcal{D}_u^h g_{N-k}\right) = \partial_v \ln\left|\frac{j_{h+1}}{j_h}\right|\partial_u\mathcal{D}_u^h g_{N-k} + \frac{j_{h+1}}{j_h}\partial_v\mathcal{D}_u^{h+1}g_{N-k} \tag{7.12}$$

we see that (7.11) reduces to the form taken by (7.9) when $h \to h+1$. Moreover, for $h = 1$, (7.9) becomes

$$\partial_v\left(\frac{j_0}{j_1}\partial_u g_{N-k}\right) + \partial_v \ln|j_1|\frac{j_0}{j_1}\partial_u g_{N-k} - g_{N-k} + \frac{j_0}{j_1}\partial_u g_{N-k-1} = 0 \tag{7.13}$$

which is just (7.8) for $i = N-k$, and is, hence, certainly true. This completes the proof of (7.9).

In the proof that the substitution sequence terminates, we need (7.9) only to derive the fact that, for $0 \leqslant k \leqslant N$,

$$g_N = \mathcal{D}_u^k g_{N-k}. \tag{7.14}$$

The proof of (7.14) is also by induction, this time over k. Assuming that the equation holds for some $k \leqslant N-1$, and writing (7.9) for the particular case $h = k+1$, we have

$$\partial_v \left(\frac{j_k}{j_{k+1}} \partial_u g_N \right) + \partial_v \ln |j_{k+1}| \frac{j_k}{j_{k+1}} \partial_u g_N - g_N + \mathcal{D}_u^{k+1} g_{N-k-1} = 0. \tag{7.15}$$

Now, (7.6) allows us to obtain

$$g_N = \mathcal{D}_u^{k+1} g_{N-k-1} \tag{7.16}$$

i.e. (7.14) for $k \to k+1$. Since, for $k = 0$, (7.14) is trivially true (and even for $k = 1$ it can be easily obtained by substituting (7.6) into (7.8) for $i = N$), its validity for $0 \leqslant k \leqslant N$ is established. In particular, we shall be interested in the case $k = N$, for which

$$g_N = \mathcal{D}_u^N g_0. \tag{7.17}$$

The equation analogous to (7.9) with $k = N$ is found by repeated differentiation of (7.7) with respect to u; we have, for $i \geqslant 1$,

$$\partial_v \mathcal{D}_u^i g_0 + \partial_v \ln |j_i| \mathcal{D}_u^i g_0 - \mathcal{D}_u^{i-1} g_0 = 0 \tag{7.18}$$

whose proof is perfectly analogous to that of (7.9). For $i = N$, (7.18) becomes, using (7.17),

$$\partial_v g_N + \partial_v \ln |j_N| g_N - \mathcal{D}_u^{N-1} g_0 = 0. \tag{7.19}$$

The final step in the proof consists in taking a further derivative with respect to u of (7.19), which gives, by (3.11), (7.6) and (7.17),

$$\frac{j_{N+1}}{j_N} g_N = 0 \tag{7.20}$$

i.e. $j_{N+1} = 0$. We have thus shown that the substitution sequence of a PW_N wave equation is upper terminating in N steps. To prove that it is also lower terminating in N steps, we could manipulate (7.1)–(7.3) to show that $j_{-N-1} = \infty$, i.e. $l_{-N-1} = 0$ by (3.17). However, it is much easier to notice that, starting with the u-normal form of the wave equation, we can repeat the proof above in a completely symmetric way to get $l_{-N-1} = 0$ directly. Hence, $\mathrm{PW}_N \Rightarrow \mathrm{KN}_N$ and, since we know already that the converse is also true, we can conclude that PW_N and KN_N are equivalent properties.

8. Conclusions and open questions

The results presented in this paper clarify the relationships between several properties of wave equations related to the absence of tails in their solutions, some of which were obvious, while some others were never explicitly analysed in the literature, at least to our knowledge. We have seen that the HP and the TF are satisfied by the same equations, and that the CPP is more general only in that it is satisfied, in addition, by special two-dimensional equations and that the PW_N propogation property and the solvability by the KN_N method for two-dimensional wave equations are equivalent. The two latter properties were not defined in more than two dimensions and they are not given in geometrical terms, despite the motivation we gave for introducing PW_N in section 3. However, we also showed that in two dimensions PW_0 is equivalent to CPP and, thus, acquires a geometrical meaning. It would therefore be interesting to investigate possible geometrical aspects of PW_N equations for higher N and whether the definition of PW_N can be meaningfully extended to higher dimensions; we will comment here on these issues.

The result of section 6, that in $1+1$ dimensions CPP is equivalent to PW_0, has the obvious corollary that no PW_N equation with $N > 0$ can satisfy CPP. This consequence is at first surprising, because from the general form (3.3) of a progressing wave, one is tempted to think that, in two dimensions at least, *all* PW_N wave equations satisfy the CPP. For, by choosing $R(u)$ and $S(v)$ to be of compact support, their derivatives will also be of compact support and all of ϕ will be made of pieces sandwiched between null coordinate lines and will, therefore, have no tails. The problem with this argument is that, although solutions with no tails *are* indeed obtained when R and S are of compact support, the latter functions are not themselves the null data which are involved in the definition of CPP; rather, the data are

$$\varphi(u) = \sum_{i=0}^{N} f_i(u,0)R^{(i)}(u) + \sum_{i=0}^{N} g_i(u,0)S^{(i)}(0) \tag{8.1}$$

$$\psi(v) = \sum_{i=0}^{N} f_i(0,v)R^{(i)}(0) + \sum_{i=0}^{N} g_i(0,v)S^{(i)}(v) \tag{8.2}$$

and the $R(u)$ and $S(v)$ that correspond to generic φ and ψ of compact support will, in general, involve integrals of φ and ψ if $N > 0$ and will thus be of non-compact support.

We can clarify this point by considering a typical example of PW_N equations, obtained by separating the angular variables in the massless Klein–Gordon equation $\Box\Phi = 0$ in four-dimensional Minkowski spacetime. Expanding Φ in spherical harmonics as $\Phi(x) = \sum_{lm} \chi_{lm}(t,r)Y_{lm}(\theta,\varphi)$ and rescaling the radial components to $\phi_{lm} := r\chi_{lm}$, we have that the ϕ_{lm} satisfy the l-dependent equation

$$\left[\partial_t^2 - \partial_r^2 + \frac{l(l+1)}{r^2}\right]\phi_{lm} \equiv \left[\partial_{uv}^2 + \frac{l(l+1)}{(v-u)^2}\right]\phi_{lm} = 0 \tag{8.3}$$

where the null coordinates u and v are defined as in (2.5) and (2.6), but now with x replaced by r; the two-dimensional equation (8.3) is PW_l and can be solved exactly [17, 20]. Since (5.23) is a necessary condition for CPP, and in the case of (8.3) we have $U = V = 0$, it follows that CPP holds only if $W = 0$; this is clearly false for $l \geqslant 1$. Let us see explicitly the failure of CPP in the simple case $l = 1$. Equation (8.3) becomes

$$\left[\partial_{uv}^2 + \frac{2}{(v-u)^2}\right]\phi(u,v) = 0 \tag{8.4}$$

which is PW_1; its general solution is

$$\phi(u,v) = \frac{2R(u)}{v-u} + R'(u) - \frac{2S(v)}{v-u} + S'(v). \tag{8.5}$$

We now look for the particular solution generated by null data with support at one point, for example,

$$\varphi(u) = \delta(u-u_0) \tag{8.6}$$

$$\psi(v) = 0. \tag{8.7}$$

These data correspond to $R(u)$ and $S(v)$ satisfying the differential equations

$$R'(u) - \frac{2}{u}R(u) + \frac{2}{u}S(0) + S'(0) = \delta(u - u_0) \tag{8.8}$$

and

$$S'(v) - \frac{2}{v}S(v) + \frac{2}{v}R(0) + R'(0) = 0 \tag{8.9}$$

whose general solutions are, respectively,

$$R(u) = \frac{u^2}{u_0^2}\theta(u - u_0) + au^2 + S'(0)u + S(0) \tag{8.10}$$

and

$$S(v) = bv^2 + R'(0)v + R(0) \tag{8.11}$$

with a and b arbitrary constants. Substituting (8.10) and (8.11) into (8.5) and using the relations $R(0) = S(0)$ and $R'(0) = S'(0)$ that follow, for example, from (8.11), we have

$$\phi(u, v) = \delta(u - u_0) + \frac{2}{u_0^2}\frac{uv}{v - u}[\theta(u - u_0) + c] \tag{8.12}$$

where $c := (a - b)u_0^2$ can be prescribed arbitrarily. It is clear from this expression that the only line $v =$ constant on which ϕ is of compact support is $v = 0$; the choice $c = 0$ gives a retarded wave, $c = -1$ an advanced wave, and any other value gives a solution which is non-zero almost everywhere on $\mathcal{M}$!

Turning now to the question of whether it is meaningful to define the class of wave equations whose general solution is a finite sum of progressing waves, in dimension $m > 2$, the simplest candidate for such an equation would again be the example $\Box\Phi = 0$ in four-dimensional Minkowski spacetime, which satisfies HP. Using the expansion in spherical harmonics given above, and (8.3), whose general solution is of the type (3.3) with $N = l$ and all the functions carrying labels l and m, we can write

$$\Phi(x) = \sum_{l=0}^{+\infty}\sum_{i=0}^{l}\sum_{m=-l}^{l}\left(r f_{ilm}(t, r)Y_{lm}(\theta, \varphi)R_{lm}^{(i)}(t - r) + r g_{ilm}(t, r)Y_{lm}(\theta, \varphi)S_{lm}^{(i)}(t + r)\right) \tag{8.13}$$

which contains derivatives up to arbitrarily high order of an infinite number of free functions R_{lm} and S_{lm} and is, therefore, not of the form (3.3). This shows that, at least in four dimensions, the definition of PW_N as it stands is empty for the case of spherical wavefronts and makes it plausible that it is ill-posed in general.

It is still useful to define a 'reduced' PW_N property for an m-dimensional equation, in the sense that all two-dimensional equations, obtained by separating out appropriately chosen coordinates, may have progressing-wave general solutions, as in the case of (8.3). A more general example, in four dimensions, would be obtained with equations of the type

$$[f(y, z)K_{t,x} + g(t, x)H_{y,z}]\phi(t, x, y, z) = 0 \tag{8.14}$$

where f and g are non-vanishing functions and $K_{t,x}$ and $H_{y,z}$ are suitable differential operators acting on the variables used as subscripts. With the separation of variables $\phi(t,x,y,z) = \psi(t,x)\varphi(y,z)$, we see that (8.14) satisfies the reduced progressing-wave propagation property if, for each value of α admitted by the equation

$$H_{y,z}\varphi(y,z) + \alpha f(y,z)\varphi(y,z) = 0 \tag{8.15}$$

the reduced wave equation

$$K_{t,x}\psi(t,x) - \alpha g(t,x)\psi(t,x) = 0 \tag{8.16}$$

is PW_N for some $N(\alpha)$. The high degree of dependence of the reduction procedure—and, hence, of the properties of the reduced equation—from specific non-covariant structure is evident. It is thus very difficult to make general statements about this property and its possible relationships with the other properties which we have discussed in this paper, independently of the explicit form of the wave equation and the choice of coordinates singled out for separation (i.e. of wave front).

It would be interesting to know whether a meaningful truly m-dimensional generalization of the PW_N property can be given for another reason as well. The expressions for the advanced or retarded Green function of $\Box\Phi = 0$ in m-dimensional Minkowski space, which for even $m \geqslant 4$ are of the form [5, 7]

$$G_m(t', \boldsymbol{x}'; t, \boldsymbol{x}) = \sum_{i=0}^{N} c_{mi}(|\boldsymbol{x}' - \boldsymbol{x}|)\delta^{(i)}(t' - t \pm |\boldsymbol{x}' - \boldsymbol{x}|) \tag{8.17}$$

with $N = (m-4)/2$, remind one of definition (3.2) and suggest a possible connection between HP and progressing waves, this time of order $N > 0$ in dimension $m > 2$. However, so far we have not been able to find such a relationship. For now, PW_N equations are left with no simple interpretation in terms of tails and no true generalization to higher dimensions, but their usefulness derives from the fact that they are exactly solvable by the method of Kundt and Newman [20], as we saw in section 7, and from their relationship to Toda lattices [25].

Taking into account all of these remarks, there is little doubt that, as far as studies of tails are concerned, the definition that should be preferred is, for its generality, the one based on HP or, equivalently, on TF. With a notion of wave tails that is now completely clear and unambiguous, one can conduct further research in several directions. Here is a list of some problems that one may address.

(i) Which are the conditions on g^{ab}, H^a and K that make (1.3) tail-free? Because of the geometrical interpretation of g_{ab} as a metric in $\mathcal{M}$, it is natural to re-state this question by asking in which spacetimes (i.e. for which class of g^{ab}) is the wave equation tail-free? This is sometimes called the *Hadamard problem* and has received some attention over the years [6, 26, 27].

(ii) What happens for vector and tensor fields? These cases include the propagation of electromagnetic and gravitational waves [15, 26, 28] and are more realistic—though somewhat more complicated—than the scalar-field case.

(iii) Which are the physical consequences and effects of wave tails? The situation that has been investigated most in detail is that of radiation from compact objects [2, 3, 8]. The corresponding problem in a cosmological background [11] has apparently been neglected, probably because of the belief that any observable effect should be extremely small. However, in this case, curvature never drops off and scattering continues forever, so it might convey a relevant amount of radiation in the interior of the light cone [9]. This immediately brings out a new problem, namely, *how much* radiation goes into the tail, which leads us to the next question.

(iv) How to quantify tails? A very natural and intuitive way would be to calculate a reflection coefficient; this is possible when backscattering is localized and there are regions in which the field is free (even asymptotically, or after a suitable coordinate transformation has been performed [8]). In a cosmological context, however, the property that makes the phenomenon potentially interesting—i.e. the fact that backscattering takes place always and everywhere—at the same time prevents us from defining purely ingoing and outgoing solutions of the wave equation and, hence, from computing reflection and transmission coefficients [9]. It is necessary to find alternative ways to quantify tails, perhaps based on the ratio between their energy content and the total energy of the field.

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